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Steady-State Matched Model Reduction for Flexible Structures with Accelerometer Measurements

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Introduction

VIBRATION control of a flexible structure may be performed with sensors that measure either displacements, rates, or accelerations at particular points. For the purposes of controller design, it is usually necessary to first carry out some form of model reduction, whereby an initially high-order finite dimensional model of the structure (for instance generated from finite element analysis) is approximated by a model of lower order. Two commonly used model reduction techniques are modal truncation, where only modes in a specified frequency range are retained and internal balancing,[1] which keeps a linear combination of the most strongly controllable and observable modes. In frequency response terms, both of these methods may be viewed as aiming to match the system response over some important part of the frequency spectrum.

It is well established[2] that the use of rate sensors that are collocated with force or torque control actuators leads to significant simplifications in the model reduction problem. In particular, the degree of observability[3] of each mode when rate sensors are used is equal to its degree of controllability. Consequently, instead of having to test both of these quantities, as is normally required for model reduction, only one test is now required. The remaining two sensor cases differ in one important respect: the high-frequency response with accelerometers approaches a finite constant value, whereas that with displacement sensors approaches zero (as a result of the inertia of the structure). In state-space terms, the fundamental difference between these cases is that the direct transmission[4] term $\boldsymbol{Du}$ in the state-space model

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{1a}$$

$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u} \tag{1b}$$

is nonzero for the system with accelerometer outputs and zero for either rate or displacement outputs.

This Note examines the role of the matrix D in the model reduction problem and, in particular, what steps must be taken to eliminate any steady-state time response error, or equivalently low-frequency frequency response error, between the full system and a reduced-order approximate model for it. (Such errors are much more important in practice than are high-frequency ones, as the latter lie outside the bandwidth of the controller.) State-space analyses of model reduction techniques are commonly given for the case of strictly proper[4] systems, that is, those with identically zero D matrices, because inertia effects dictate that most physical systems are of this class. Consequently, the connection between D and model reduction is not typically investigated. Furthermore, the degrees of controllability and observability of structural modes (the basis of model reduction by modal truncation), and the controllability and observability Grammian matrices of the system (the basis of model reduction by internal balancing), are all independent of the D matrix. These conditions, together with the invariance of D under any nonsingular state transformation, that is, any change in the set of basis vectors used to describe state space, may lead to the erroneous conclusion that D should simply be left unchanged when computing a reduced-order model for a flexible structure with accelerometer measurements. However, this would actually give rise to significant steady-state errors in the response of the reduced model. The Note details how to avoid this difficulty when a reduced-order model is generated by means of any general model reduction technique, by suitably constructing the D matrix for the reduced model. Details are then given for the particular cases of model reduction by the use of either modal truncation or internal balancing. Finally, these conclusions are illustrated by application to a simple numerical example.

Problem Formulation

Consider an n-mode model (generated, for instance, by means of a finite element analysis) for the structural dynamics of a modally damped, nongyroscopic, noncirculatory flexible structure controlled by $m(<n)$ actuators and $p(<n)$ sensors. This model can be written in modal form[5] as

$$\ddot{\boldsymbol{\eta}} + \mathrm{diag}\,(2\zeta_i\omega_i)\dot{\boldsymbol{\eta}} + \mathrm{diag}\left(\omega_i^2\right)\boldsymbol{\eta} = \boldsymbol{\Phi}_a^T\boldsymbol{u} \tag{2a}$$

$$\boldsymbol{y} = \boldsymbol{\Phi}_{sa}\ddot{\boldsymbol{\eta}} + \boldsymbol{\Phi}_{sr}\dot{\boldsymbol{\eta}} + \boldsymbol{\Phi}_{sd}\boldsymbol{\eta} \tag{2b}$$

where $\boldsymbol{\eta}$ is the vector of model coordinates and ω_i and ζ_i are the natural frequency and damping ratio of the ith mode, respectively. For typical flexible structures,[6] the $\{\zeta_i\}$ are quite low, for example, 0.005, and the $\{\omega_i\}$ often occur in clusters of nearly repeated frequencies in flexible spacecraft applications. The modal influence matrices $\boldsymbol{\Phi}_a$, $\boldsymbol{\Phi}_{sa}$, $\boldsymbol{\Phi}_{sr}$, and $\boldsymbol{\Phi}_{sd}$ are evaluated at the actuator, acceleration sensor, rate sensor, and displacement sensor stations, respectively. Thus, $\boldsymbol{\Phi}_a(i, j)$ is the amplitude of mode shape j evaluated at actuator station i, etc. The output $\mathbf{y}$ will be assumed to consist exclusively of either acceleration, rate, or displacement measurements, and not a mixture. Two of the matrices $\boldsymbol{\Phi}_{sa}$, $\boldsymbol{\Phi}_{sr}$, and $\boldsymbol{\Phi}_{sd}$ will, therefore, always be taken to be zero here. (This limitation is purely for clarity of presentation; no serious technical issues arise for systems with output vectors made up of a combination of displacements, rates, and/or accelerations.)

The structural model given by Eq. (2) is clearly finite dimensional, although the number of modes it contains, n, may be very high. Note, therefore, that this model is itself only an approximation to the dynamics of the physical structure that is being studied: This structure is inherently[5] an infinite dimensional, continuous system.

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Consequently, although Eq. (2) can be thought of as the original or full-order model in the context of this Note, it does not exactly describe the dynamics of the true physical structure. In particular, truncated modal models neglect the contribution of the omitted higher modes to the flexibility of the structure. This effect, termed residual flexibility, is important even at low frequency. The interested reader is referred to Refs. 5 and 7 for a clear analysis of this question.

Note that the focus of the model reduction results that will be presented here is on the reduction of the number of flexible modes that are required to model the dynamics of the system adequately. Because the original structural model may well possess hundreds of flexible modes, there is great scope for achievement of a significant reduction in total model order by operation on these modes. If the system possessed rigid-body modes, there can be at most six of these; they, therefore, offer very limited possibilities for model reduction. In fact, the safest approach would be simply to retain in the reduced-order model any rigid-body modes possessed by the original model. In light of these facts, the methods presented in this paper focus exclusively on flexible structures with no rigid-body modes. It seems likely that the results obtained can be generalized to apply to flexible structures that do possess rigid-body modes, but the details will not be given in this Note.

State-Space Structural Models

Definition of the state vector $\boldsymbol{x} = (\dot{\eta}_1, \omega_1\eta_1, \ldots, \dot{\eta}_n, \omega_n\eta_n)^T$ for the structural model Eq. (2) (assumed to possess no rigid-body modes) yields a state-space model of the form of Eq. (1) for it. This state-space model has matrices $A = \text{blkdiag}\,(A_i)$ and $B = (B_1^T, \ldots, B_n^T)^T$, where

$$A_i = \begin{pmatrix} -2\zeta_i\omega_i & -\omega_i \\ \omega_i & 0 \end{pmatrix}, \qquad B_i = \begin{pmatrix} \varphi_{a_i}^T \\ \mathbf{0} \end{pmatrix} \tag{3}$$

and where φ_{a_i} is the ith column of $\boldsymbol{\Phi}_a$.

The form of the matrices C and D depends on whether displacement, rate, or acceleration outputs are considered. Of the three cases, the most complicated is that of acceleration measurements because accelerations are not state variables of the system. However, Eq. (2a) can be used to put this type of output into standard state-space form also, by expressing accelerations in terms of displacements and rates, which are state variables. The resulting output equation is $\boldsymbol{\Phi}_{sa}\ddot{\boldsymbol{\eta}} = -\boldsymbol{\Phi}_{sa}\,\text{diag}\,(2\zeta_i\omega_i)\dot{\boldsymbol{\eta}} - \boldsymbol{\Phi}_{sa}\,\text{diag}\,(\omega_i^2)\boldsymbol{\eta} + \boldsymbol{\Phi}_{sa}\boldsymbol{\Phi}_a^T\boldsymbol{u}$. The matrices $C = (C_1, \ldots, C_n)$ and D that describe the output equation are then given as follows for the three possible cases:

For displacement outputs,

$$C_i = \begin{pmatrix} \mathbf{0} & \omega_i^{-1}\varphi_{sd_i} \end{pmatrix}, \qquad D = 0 \tag{4a}$$

for rate outputs

$$C_i = \begin{pmatrix} \varphi_{sr_i} & \mathbf{0} \end{pmatrix}, \qquad D = 0 \tag{4b}$$

and for acceleration outputs

$$C_i = \begin{pmatrix} -2\zeta_i\omega_i\varphi_{sa_i} & -\omega_i\varphi_{sa_i} \end{pmatrix}, \qquad D = \boldsymbol{\Phi}_{sa}\boldsymbol{\Phi}_a^T \tag{4c}$$

where φ_{sa_i}, φ_{sr_i}, and φ_{sd_i} are the ith columns of $\boldsymbol{\Phi}_{sa}$, $\boldsymbol{\Phi}_{sr}$, and $\boldsymbol{\Phi}_{sd}$, respectively. Note that the accelerometer case is the only one for which the direct transmission term D is nonzero. This case will be the focus of the text that follows.

A general state transformation $\tilde{\boldsymbol{x}} = T\boldsymbol{x}$, where the matrix T is square and nonsingular, corresponds to a change in the basis vectors that are used to parameterize state space. The effect of such a change in basis vectors on the state-space model $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, $\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u}$ is to transform it to a new model:

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}\boldsymbol{u} \tag{5a}$$

$$\boldsymbol{y} = \tilde{C}\tilde{\boldsymbol{x}} + \tilde{D}\boldsymbol{u} \tag{5b}$$

where $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$, and $\tilde{D} = D$. It can be observed that the direct transmission matrix D is unaffected by any state transformation of this form. Consequently, little attention is typically paid to this matrix in discussions of state transformations in the controls literature. In addition, because many model reduction procedures (notably internal balancing[1]) include some form of state transformation, it may appear that D will not be altered in the course of model reduction. However, this is not actually the case, as will now be demonstrated.

Model Reduction for Flexible Structures

Model reduction procedures for state-space models of flexible structures can, without loss of generality, be formulated to consist of the following two steps: 1) application of a state transformation T, that is, change in state-space basis vectors as just described, and 2) truncation of the transformed state-space model [Eq. (5)], that is, a discarding of $2(n-r)$ of the states of the transformed model. This truncation is based on some measure, such as the natural frequencies of the structure, or the degrees of controllability and observability, or the component costs of its modes.

The resulting reduced-order model then has state-space representation

$$\dot{\hat{\boldsymbol{x}}} = \hat{A}\hat{\boldsymbol{x}} + \hat{B}\boldsymbol{u} \tag{6a}$$

$$\hat{\boldsymbol{y}} = \hat{C}\hat{\boldsymbol{x}} + \hat{D}\boldsymbol{u} \tag{6b}$$

with matrices $\hat{A} = T_1AS_1$, $\hat{B} = T_1B$, and $\hat{C} = CS_1$ ($\hat{D}$ discussed shortly), where the rectangular matrices T_1 and S_1 are given as

$$T_1 = (I_{2r} \quad 0)T \tag{7a}$$

$$S_1 = T^{-1}\begin{pmatrix} I_{2r} \\ 0 \end{pmatrix} \tag{7b}$$

These rectangular matrices describe not only the initial full state basis change described by T (step 1), but also the truncation carried out in step 2. [Note that, when very large structural models are dealt with in practice, it would be inefficient actually to compute the full transformed state-space model of Eq. (5) and then truncate it. This would require generation of a transformed n-state model, only to discard the bulk of the model immediately afterward. Consequently, a practical procedure would ideally not actually be implemented exactly as described by steps 1 and 2. However, this approach was taken for clarity of exposition and does not affect the model reduction results that will be obtained in any way.]

If the reduced-order model equation (6) is to give the same steady-state output as the original model [Eqs. (3) and (4)], not only must the matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ be as given earlier, but the direct transmission matrix $\hat{D}$ must also be generated correctly. The following result describes how to achieve this.

Lemma: The direct transmission matrix $\hat{D}$ that corresponds to the reduced-order model described by Eq. (6) for a flexible structure with accelerometer measurements is given as

$$\hat{D} = \hat{C}\hat{A}^{-1}\hat{B} \tag{8}$$

Proof: In the steady-state step response of a flexible structure, $\dot{\boldsymbol{\eta}} \to \mathbf{0}$ and $\ddot{\boldsymbol{\eta}} \to \mathbf{0}$. Consequently, the state derivative $\dot{\boldsymbol{x}} \to \mathbf{0}$ and, if all outputs are generated by accelerometers, $\boldsymbol{y} \to \mathbf{0}$ also. However, the reduced-order state-space model yields $\dot{\hat{\boldsymbol{x}}} = \hat{A}\hat{\boldsymbol{x}} + \hat{B}\boldsymbol{u} \to \mathbf{0}$, or $\hat{\boldsymbol{x}} \to -\hat{A}^{-1}\hat{B}\boldsymbol{u}_{ss}$. The output equation, thus, becomes $\boldsymbol{y} = \hat{C}\hat{\boldsymbol{x}} + \hat{D}\boldsymbol{u} \to [\hat{D} - \hat{C}\hat{A}^{-1}\hat{B}]\boldsymbol{u}_{ss}$, which must tend to zero for any choice of step input vector $\boldsymbol{u}_{ss}$. $\hat{D}$ must, therefore, be as given by Eq. (8). QED

The model reduction procedure that is summarized by Eqs. (6–8) is fully general. The differences between the various procedures that are used lie in two areas: the choice of initial start basis change matrix T, and the criterion used to either retain or discard any particular transformed state in the final reduced model. Two particular examples of structural model reduction procedures will now be analyzed in the context of this general formulation, namely, modal truncation and internal balancing.

Modal Truncation

In model truncation, certain structural modes of the original model Eq. (2) are discarded. Because these modes correspond directly to states of the state-space model given by Eqs. (3) and (4), no preliminary change of state-space basis is required, that is, $T = I_{2n}$ in step 1. Thus, if the first r modes are to be retained in the reduced-order model, $T_1 = S_1^T = (I_{2r}\ \ 0)$ in Eq. (7) of step 2. The resulting reduced-order direct transmission matrix, from Eq. (8), is then

$$\begin{aligned}\hat{D} &= \hat{C}\hat{A}^{-1}\hat{B} = \sum_{i=1}^{r} C_i A_i^{-1} B_i \\ &= \sum_{i=1}^{r} \begin{pmatrix} -2\zeta_i\omega_i\varphi_{sa_i} & -\omega_i\varphi_{sa_i}\end{pmatrix}\begin{pmatrix} 0 & \omega_i^{-1} \\ -\omega_i^{-1} & -2\zeta_i\omega_i^{-1}\end{pmatrix}\begin{pmatrix}\varphi_{a_i}^T \\ \mathbf{0}\end{pmatrix} \\ &= \sum_{i=1}^{r}\begin{pmatrix} -2\zeta_i\varphi_{sa_i} & -\varphi_{sa_i}\end{pmatrix}\begin{pmatrix}\mathbf{0} \\ -\varphi_{a_i}^T\end{pmatrix} \\ &= \sum_{i=1}^{r}\varphi_{sa_i}\varphi_{a_i}^T = \mathbf{\Phi}_{sa}\begin{pmatrix} I_r & 0 \\ 0 & 0\end{pmatrix}\mathbf{\Phi}_a^T \end{aligned} \tag{9}$$

Modal truncation can equivalently be regarded as an omission of those columns of the modal influence matrices in Eq. (2) that correspond to modes that are to be discarded. If r modes are again retained, the relevant reduced-order modal influence matrices become

$$\hat{\mathbf{\Phi}}_a = \mathbf{\Phi}_a\begin{pmatrix} I_r \\ 0\end{pmatrix} \tag{10a}$$

$$\hat{\mathbf{\Phi}}_{sa} = \mathbf{\Phi}_{sa}\begin{pmatrix} I_r \\ 0\end{pmatrix} \tag{10b}$$

The resulting direct transmission matrix, computed by analogy with the expression [Eq. (4c)] that was used for the original system, is then

$$\hat{D} = \hat{\mathbf{\Phi}}_{sa}\hat{\mathbf{\Phi}}_a^T = \mathbf{\Phi}_{sa}\begin{pmatrix} I_r & 0 \\ 0 & 0\end{pmatrix}\mathbf{\Phi}_a^T$$

or precisely as given earlier by the general expression Eq. (8). Also note that, if the original D were used in the reduced-order model instead of $\hat{D}$, a steady-state error in response to step inputs(s) of

$$\delta\boldsymbol{y} = (D - \hat{D})\boldsymbol{u}_{ss} = \mathbf{\Phi}_{sa}\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r}\end{pmatrix}\mathbf{\Phi}_a^T\boldsymbol{u}_{ss} \tag{11}$$

would be obtained, where $\boldsymbol{u}_{ss} \equiv \boldsymbol{u}(t)$, $t > 0$, that is, its ith entry is 1 if a unit step is applied at actuator i, and zero if not.

Internal Balancing

A particular case of state transformation that is often used in model reduction is internal balancing.[1] The controllability and observability Grammians of the state-space model $\{A, B, C, D\}$, denoted by W_c and W_o, respectively, are the solutions of the algebraic Lyapunov equations

$$AW_c + W_cA^T + BB^T = 0 \tag{12}$$

$$A^TW_o + W_oA + C^TC = 0 \tag{13}$$

[The block diagonal form of Eq. (3) for A corresponding to a flexible structure can be exploited[3,8,9] to give closed-form solutions for these equations; these are considerably more efficient to compute than is the use of a general-purpose Grammian routine.] The essence of internal balancing is that a unique state transformation matrix T_b always exists,[1] for which the transformed state-space model $\{T_bAT_b^{-1}, T_bB, CT_b^{-1}, D\}$, that is, given by Eq. (5) with $T = T_b$, has diagonal and equal controllability and observability Grammians:

$$\bar{W}_c = \bar{W}_o = \Sigma \equiv \mathrm{diag}(\sigma_i) \tag{14}$$

where the Hankel singular values $\{\sigma_i\}$ are ordered such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Computation of the transformation matrix T_b can be shown[10] to amount to the solution of a standard eigenproblem. Note that, for a modal model with widely separated natural frequencies, it can be shown[2,11] that $T_b \approx I$: The original modal model is approximately balanced. However, this is not true for structures with closely spaced frequencies,[12] as is often the case in practical spacecraft applications.

The Hankel singular values lead to a simple procedure to obtain a reduced-order approximate model for the original system: Delete those balanced states corresponding to all $\{\sigma_i\}$ below some specified threshold. The resulting dominant reduced-order model will match the full system with an accuracy related to the sizes of those Hankel singular values that were discarded, thus, giving a guideline for the selection of an acceptable reduced-model order n_r. (See Ref. 1 for further details.) However, previous studies of model reduction by the use of balancing for flexible structures did not specifically address the question of steady-state errors for the case of accelerometer measurements. Equation (8) now shows how these can be eliminated.

Simple Numerical Example

For illustration of the preceding model reduction results, consider a simple five-mode model for a uniform cantilever beam with dimensions taken to be length 5 m, width 0.1 m, and depth 0.01 m and constructed of aluminum (density 2.7×10^3 kg/m^3 and Young's modulus 7.0×10^{10} N/m^2). A single force actuator is located at the free tip of the beam, a liner accelerometer collocated with it, and a damping ratio of 1% is assumed to apply for each structural mode. Frequency-based modal truncation, that is, discarding the highest-frequency modes is then used to obtain reduced order models of order 2,3, and 4 for this structure, giving a set of matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ corresponding to Eq. (6) for each reduced model order r.

Figure 1 compares the frequency response of the full-order system (solid curve) with the responses produced by these reduced-order models if the D matrix is either 1) erroneously left equal to that of the

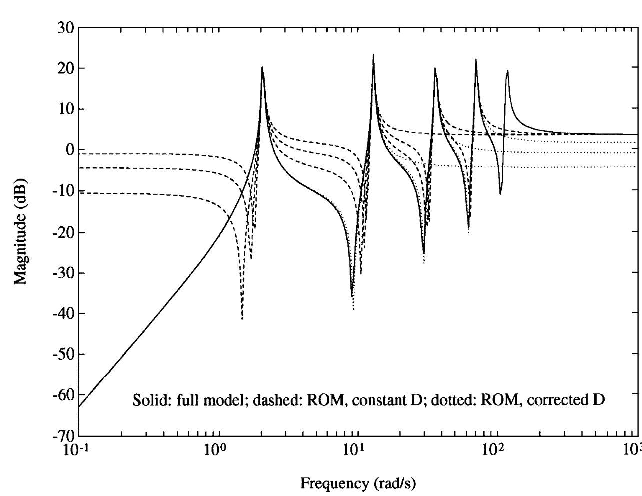

Fig. 1 Acceleration frequency responses: ——, full system and reduced models of various order using direct transmission matrix - - - -, D or · · · ·, $\hat{D}$.

five-mode model (dashed curves) or 2) replaced by $\hat{D}$ as given from Eq. (9) (dotted curves). The presence of sizable biases in the dashed responses at low frequencies, leading to significant steady-state errors, can clearly be seen. By contrast, the dotted curves obtained by the use of $\hat{D}$ can be seen to overlay the solid curves at all frequencies below 4 Hz, thus, achieving the desired absence of steady-state errors.

Finally, note that very similar results would have been obtained if balancing were used to perform the reduction of this particular system. This follows from the fact that all natural frequencies of the beam are widely spaced, thus, putting this system in the class of structures for which the modal model is approximately balanced.

Conclusions

Any state-space model for a flexible structure that is to be controlled by means of accelerometer measurements will possess a nonzero direct transmission term, that is, a nonzero D matrix. This Note has examined the role of this matrix in the model reduction problem. In particular, the D matrix corresponding to the reduced-order model cannot simply be taken as equal to that of the original system; if this were done,the steady-state output of the reduced model would possess an appreciable error. It was shown how to avoid this difficulty by computation of the correct $\hat{D}$ for a reduced-order model that is computed by any general model reduction technique. Details were then given for the two common model reduction approaches of modal truncation and internal balancing. Finally, these conclusions were illustrated by application to a simple numerical example.

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